Math 2040 C Week 8

Diagonalization of linear operator

Defn 5.36 Let $T \in L(V)$, $\lambda \in \mathbb{F}$.

The eigenspace of T corresponding to λ is defined to be

$$E(\lambda, T) = \text{null}(T - \lambda I_V)$$

Rmk

① $E(\lambda, T)$ is a subspace of V

② $E(\lambda, T) \neq \{0\} \iff \lambda$ is an e.value

③ If λ is eigenvalue,

$$E(\lambda, T) = \{\text{e.vector corr to } \lambda\} \cup \{\vec{0}\}$$

Prop 5.38 Let $T \in L(V)$.

If $\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of T. Then

① $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum

② If $\dim V < \infty$, then

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V$$

Pf ① Suppose $v_i \in E(\lambda_i, T)$ for $i = 1, \cdots, m$ such that $v_1 + v_2 + \cdots + v_m = \vec{0}$.

Let $S = \{i : v_i \neq \vec{0}\}$.

If $S \neq \phi$, then $\{v_i : i \in S\} \subseteq V$ is an non-empty subset of eigenvectors

By Prop 5.10, $\{v_i : i \in S\}$ is linearly independent

However, $\sum_{i \in S} v_i = \sum_{i=1}^{m} v_i = \vec{0}$, a contradiction

$\therefore S = \phi$ and $v_i = \vec{0}$ $\forall i = 1, \cdots, m$

② $\bigoplus_{i=1}^{m} E(\lambda_i, T) \leq V$.

$\therefore \sum_{i=1}^{m} \dim E(\lambda_i, T) = \dim\left(\bigoplus_{i=1}^{m} E(\lambda_i, T)\right)$

$\leq \dim V$

Defn 5.39 Let $T \in L(V)$.

T is called diagonalizable if $\exists$ ordered basis α of V s.t. $M(T, \alpha)$ is diagonal

eg. Let $T: \mathbb{C}^2 \to \mathbb{C}^2$, $T(z,w) = (-w, z)$

Then $T(1,-i) = (i, 1) = i(1,-i)$

$T(1,i) = (-i, 1) = -i(1,i)$

$\therefore M(T, \{(1,-i),(1,i)\}) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

and T is diagonalizable over $\mathbb{C}$

Prop 5.41 Let $\dim V < \infty$, $T \in L(V)$, $\lambda_1, \cdots, \lambda_m$ are all the distinct eigenvalues of T. Then TFAE:

ⓐ T is diagonalizable

ⓑ T has an eigenbasis

ⓒ $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$

ⓓ $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$

Pf (a $\Rightarrow$ b) Suppose T is diagonalizable, then $\exists$ ordered basis $\alpha = \{v_1, \cdots, v_n\}$ of V s.t.

$$M(T, \alpha) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \text{ is diagonal}$$

Then for each $i = 1, \cdots, n$, $T(v_i) = \lambda_i v_i$

Also, α is basis $\Rightarrow v_i \neq \vec{0}$

$\therefore v_i$ is an e.vector

$\therefore$ The basis α is an e.basis of T

$(b \Rightarrow c)$

Suppose $\alpha = \{v_1, \cdots, v_n\}$ is an e.basis of T

By reordering if necessary, can assume

$\exists\ 0 = r_0 < r_1 < \cdots < r_{m-1} < r_m = n$ s.t.

$v_{r_{j-1}+1}, \cdots, v_{r_j}$ are e.vectors corr. to λ_j

For any $v \in V$, $\exists\ c_1, \cdots, c_n$ s.t.

$$v = \sum_{i=1}^{n} c_i v_i$$

For $j = 1, \cdots, m$, let $w_j = \sum_{k=r_{j-1}+1}^{r_j} c_k v_k$

Then $w_j \in E(\lambda_j, T)$ and $v = \sum_{j=1}^{m} w_j$

Hence, $V = E(\lambda_1, T) + \cdots + E(\lambda_m, T)$

$= E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$

by Prop 5.38.

$(c \Leftrightarrow d)$ By 5.38,

$$\sum_{j=1}^{m} \dim E(\lambda_j, T) = \dim\left(\bigoplus_{j=1}^{m} E(\lambda_j, T)\right) \le \dim V \text{ and}$$

equality holds (i.e. ⓒ) $\Leftrightarrow \bigoplus_{j=1}^{m} E(\lambda_j, T) = V$ (i.e. ⓓ)

$(c \Rightarrow a)$ Suppose $V = \bigoplus_{j=1} E(\lambda_j, T)$

let $\beta_j = \{v_{j1}, \cdots v_{jn_j}\}$ be an ordered basis of $E(\lambda_j, T)$

and $\beta = \beta_1 \cup \cdots \cup \beta_m$

$E(\lambda_j, T) = \operatorname{span} \beta_j$ and $V = \bigoplus_{j=1}^{m} E(\lambda_j, T) \Rightarrow V = \operatorname{span} \beta$

Also, we proved $c \Leftrightarrow d$,

$\therefore\ |\beta| \le \sum |\beta_j| = \sum \dim E(\lambda_j, T) \overset{ⓓ}{=} \dim V$

Hence, β is a basis of V.

Note $\forall v \in \beta = \beta_1 \cup \cdots \cup \beta_m$, $T(v) = \lambda_j v$ for some $j = 1, \cdots m$

$\therefore\ M(T, \beta)$ is a diagonal matrix.

eg. Consider $D \in L(P_n(\mathbb{R}))$, $Dp = p'$

Suppose $Dp = \alpha p$, $p \neq \vec{0}$

If $\deg p \geq 1$ and $\alpha \neq 0$, then

$\deg Dp = \deg p - 1 \neq \deg \alpha p$

$\therefore \deg p = 0$ or $\alpha = 0$

If $\deg p = 0$, then

$Dp = \vec{0} \Rightarrow \alpha = 0$

$\therefore \alpha = 0$ is the only eigenvalue

Clearly, $E(T, 0) = \operatorname{span}\{1\}$

and $\dim E(T, 0) = 1$

$\therefore D$ is diagonalizable

$\Leftrightarrow \dim P_n(\mathbb{R}) = 1$

$\Leftrightarrow n = 0$

Prop 5.44 Let $\dim V = n$. If $T \in L(V)$ has n distinct eigenvalues, then T is diagonalizable

Pf Let $\lambda_1, \cdots, \lambda_n$ be the distinct e.values of T and $v_1, \cdots, v_n$ be corresponding e.vectors.

By Prop 5.10,

$v_1, \cdots, v_n$ are n lin. indept. eigenvectors

$\Rightarrow \{v_1, \cdots, v_n\}$ is an eigenbasis

$\Rightarrow T$ is diagonalizable

Alternative Pf λ_i are e.values $\Rightarrow \dim E(\lambda_i, T) \geq 1$

$$\therefore n \underset{5.38}{\geq} \sum_{i=1}^{n} \dim E(\lambda_i, T) \geq n$$

$$\Rightarrow \sum_{i=1}^{n} \dim E(\lambda_i, T) = n \Rightarrow T \text{ is diagonalizable}$$

Determine diagonalizability of operator using matrix

eg Are the following $T \in \mathcal{L}(P_2(\mathbb{R}))$ diagonalizable? If so, find eigenbasis.

ⓐ $T(p(x)) = (x-1)p'(x) + p(1)$

Let $\beta = \{1, x, x^2\}$

Then $T(1) = 1 \qquad T(x) = x$

$T(x^2) = 1 - 2x^2 + x^2$

Let $A = M(T,\beta) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$

Characteristic polynomial of A is

$p(t) = \det(A - tI_3)$

$= \begin{vmatrix} 1-t & 0 & 1 \\ 0 & 1-t & -2 \\ 0 & 0 & 2-t \end{vmatrix}$

Another way: A is upper triangular $\Downarrow$ 5.32

$= -(1-t)^2(2-t)$

$\therefore$ eigenvalues of A and T are $\lambda_1 = 1, \lambda_2 = 2$

For $\lambda_1 = 1$, $A - \lambda_1 I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\uparrow \ \uparrow$ free

Let $c_1 = s$, $c_2 = t$, then

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Basis of $E(1,A)$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Similarly, for $\lambda_2 = 2$, $A - \lambda_2 I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Basis of $E(2,A)$: $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

$\dim E(1,A) + \dim E(2,A) = 2 + 1 = 3$

$\Rightarrow$ A is diagonalizable

$\therefore$ T is also diagonalizable, with eigenbasis $\alpha = \{1, x, 1-2x+x^2\}$

$M(T,\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Rmk $D = Q^{-1}AQ$, where

$$D = M(T,\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$Q = M(I,\alpha,\beta) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

b. $T(p(x)) = (x-1)p'(x) + p(2)$

let $\beta = \{1, x, x^2\}$

$$A = M(T,\beta) = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

which has eigenvalue $\lambda_1 = 1$, $\lambda_2 = 2$

For $\lambda_1 = 1$, $A - \lambda_1 I = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Basis of $E(1,A)$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

For $\lambda_2 = 2$, $A - \lambda_2 I_1 = \begin{bmatrix} -1 & 1 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Basis of $E(2,A)$: $\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$

$\therefore \dim E(1,A) + \dim E(2,A) = 2 < 3$

$\therefore$ A is not diagonalizable $\Rightarrow$ T is not diagonalizable

c. $T(p(x)) = (x-1)p'(x)$

$M(T,\beta) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$ has 3 distinct eigenvalues $0, 1, 2$

By 5.44, T is diagonalizable

By computing eigenvectors, we find T has eigenbasis $\alpha = \{1, x, 1-2x+x^2\}$ and

$$M(T,\alpha) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Inner Product Space

- In $\mathbb{R}^n$,

  Dot product $\rightsquigarrow$ length, angle, perpendicular, projection

  $\vec{a}\cdot\vec{b} = a_1b_1 + \cdots + a_nb_n$

- In $\mathbb{C}$, $z = x + yi$, $x, y \in \mathbb{R}$

  length of $z = |z| = \sqrt{z\bar{z}}$

  $\Big\downarrow$ Generalize

Inner product structure on vector space over $\underbrace{\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}}_{\text{a must for inner product}}$

Defn 6.3, 6.5 Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Let V be a vector space $/\mathbb{F}$. An inner product on V is a function
$$V \times V \longrightarrow \mathbb{F}$$
$$(u, v) \longmapsto \langle u, v\rangle$$
such that

$\left.\begin{array}{l}\text{(IP1) Additivity in 1st slot} \\ \langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle \quad \forall u, v, w \in V \\ \text{(IP2) Homogenity in 1st slot} \\ \langle \lambda u, v\rangle = \lambda\langle u, v\rangle \quad \forall u, v \in V,\ \lambda \in \mathbb{F}\end{array}\right\}$ linear in 1st slot

(IP3) Conjugate symmetry

$\langle u, v\rangle = \overline{\langle v, u\rangle} \quad \forall u, v \in V$

$\left.\begin{array}{l}\text{(IP4) Positivity (Rmk } \langle v,v\rangle \in \mathbb{R} \text{ by IP3)} \\ \langle v, v\rangle \geq 0 \quad \forall v \in V \\ \text{(IP5) Definiteness} \\ \langle v, v\rangle = 0 \iff v = 0\end{array}\right\}$ Positive definite

An inner product space is a vector space V along with an inner product on V

Examples of Inner product spaces

① Euclidean inner product on $\mathbb{F}^n$

Let $x=(x_1,\cdots,x_n), y=(y_1,\cdots,y_n)\in\mathbb{F}^n$

Define $\langle x,y\rangle=\sum_{i=1}^{n}x_i\overline{y_i}$

$*$ Variation: Let $c_1,\cdots,c_n>0$ and

define $\langle x,y\rangle=\sum_{i=1}^{n}c_i x_i\overline{y_i}$

weighted

② Let $a,b\in\mathbb{R}$, $a<b$

$V=C([a,b])$

$=\{f:[a,b]\to\mathbb{R} : f \text{ is continuous}\}$

Define

$$\langle f,g\rangle=\int_a^b f(x)g(x)\,dx$$

for $f,g\in V$

- Variation 1: Replace $\mathbb{R}$ by $\mathbb{C}$ and define

$$\langle f,g\rangle=\int_a^b f(x)\overline{g(x)}\,dx$$

e.g. $a=0$, $b=2\pi$, $f(x)=e^{ix}=\cos x+i\sin x$

then $\langle f,f\rangle=\int_0^{2\pi}e^{ix}\cdot\overline{e^{ix}}\,dx=\int_0^{2\pi}1\,dx=2\pi$

- Variation 2: Replace V by $P(\mathbb{F})\subseteq C([a,b])$

Q Can we define inner product as follows?

ⓐ $\langle(x_1,x_2),(y_1,y_2)\rangle=x_1y_2+x_2y_1$ on $\mathbb{R}^2$

ⓑ $\langle p,q\rangle=\int_{-\infty}^{\infty}p(x)q(x)\,dx$ on $P_3(\mathbb{R})$

ⓒ $\left\langle\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix},\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}\right\rangle=\sum_{1\le i,j\le 2}a_{ij}b_{ij}$ on $M_{2\times2}(\mathbb{C})$

Ans No! ⓐ Not positive definite

ⓑ The integral diverges $\Rightarrow\langle p,q\rangle\notin\mathbb{R}$

ⓒ Not conjugate symmetric

Prop 6.7 Let V be inner product space,

a. For fixed $u \in V$, $\varphi : V \to \mathbb{F}$ defined by

$$\varphi(v) = \langle v, u \rangle \text{ is linear}$$

variable fixed

b. $\langle \vec{0}, v \rangle = 0$, $\langle v, \vec{0} \rangle = 0 \quad \forall v \in V$

c. $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

d. $\langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle$

$\Big\}$ Conjugate linear in 2nd slot

Pf b. $\langle \vec{0}, v \rangle = \langle 0 \cdot \vec{0}, v \rangle = 0 \langle \vec{0}, v \rangle = 0$

Similar for $\langle v, \vec{0} \rangle$

d.
$$\begin{aligned} \langle u, \lambda v \rangle &= \overline{\langle \lambda v, u \rangle} \\ &= \overline{\lambda \langle v, u \rangle} \\ &= \bar{\lambda}\, \overline{\langle v, u \rangle} \\ &= \bar{\lambda} \langle u, v \rangle \end{aligned}$$

Defn 6.8, 6.11 Let V be inner product space

① For $v \in V$, the norm of v is defined to be

$$\|v\| = \sqrt{\langle v, v \rangle}$$

② If $u, v \in V$ and $\langle u, v \rangle = 0$, then u, v are called orthogonal, denoted by $u \perp v$

eg Consider $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$ on $P(\mathbb{R})$. Then

$$\langle x^m, x^n \rangle = \int_{-1}^{1} x^m \cdot x^n \, dx = \begin{cases} \dfrac{2}{m+n+1} & \text{if } m+n \text{ is even} \\ 0 & \text{if } m+n \text{ is odd} \end{cases}$$

$$\therefore \|x^m\| = \sqrt{\langle x^m, x^m \rangle} = \sqrt{\frac{2}{2m+1}}$$

$x^m \perp x^n$ if m is odd, n is even

Prop 6.10, 6.12 For v in an inner product space

① $\|v\| = 0 \iff v = \vec{0}$ ② $\|\lambda v\| = |\lambda| \, \|v\|$

③ $\vec{0} \perp v$ ④ $v \perp v \iff v = \vec{0}$

Let V be an inner product space

Prop 6.13 (Pythagorean Theorem)

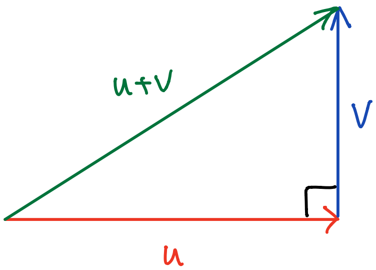
Suppose $u \perp v$. Then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Pf

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \langle u, u+v\rangle + \langle v, u+v\rangle \\
&= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\
&= \|u\|^2 + \|v\|^2
\end{aligned}$$

$= 0 \because u \perp v$

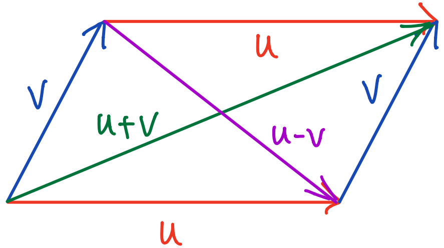


Prop 6.22 (Parallelogram Equality)

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

Pf Exercise

Geometric Intepretation:



In a parallelogram,

Sum of squares of lengths of 2 diagonals	=	Sum of squares of lengths of 4 sides

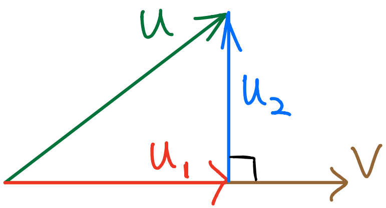
Orthogonal Decomposition

Given $u, v \in V$, $v \neq \vec{0}$

Want to decompose $u = u_1 + u_2$

s.t. (1) $u_1 = cv$, $c \in \mathbb{F}$

(2) $u_2 \perp v$



Note (2) $\Leftrightarrow$ $0 = \langle u_2, v \rangle$

$= \langle u - u_1, v \rangle$

$= \langle u - cv, v \rangle$

$= \langle u, v \rangle - c\|v\|^2$

$\Leftrightarrow$ $c = \dfrac{\langle u, v \rangle}{\|v\|^2}$

<u>Prop 6.14</u> Suppose $u, v \in V$, $v \neq \vec{0}$.

Let $c = \dfrac{\langle u, v \rangle}{\|v\|^2}$ and $w = u - cv$

Then $u = cv + w$ with $w \perp v$

<u>Rmk</u> cv is called the <u>orthogonal projection</u> of u onto v, denoted by $\boxed{P_v u = \dfrac{\langle u, v \rangle}{\|v\|^2} v}$

<u>eg</u> Let $v = \left(\frac{3}{5}, \frac{4}{5}\right)$

Then $P_v e_1 = \dfrac{\langle e_1, v \rangle}{\|v\|^2} v = \dfrac{\frac{3}{5}}{1^2} v = \left(\frac{9}{25}, \frac{12}{25}\right)$

$P_v e_2 = \dfrac{\langle e_2, v \rangle}{\|v\|^2} v = \dfrac{\frac{4}{5}}{1^2} v = \left(\frac{12}{25}, \frac{16}{25}\right)$

Hence, for $P_v \in \mathcal{L}(\mathbb{R}^2)$ and $\beta = \{e_1, e_2\}$

$$M(P_v, \beta) = \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$$

$$\boxed{\begin{array}{l}\underline{\text{Prop 6.15}}\ \text{(Cauchy Schwarz Inequality)} \\ |\langle u,v\rangle| \leq \|u\|\|v\| \quad \text{for any } u,v \in V \\ \text{Equality holds } |\langle u,v\rangle| = \|u\|\|v\| \iff u = cv \text{ or } v = cu \text{ for some } c \in \mathbb{F}\end{array}}$$

<u>Pf</u> If $v=0$, L.H.S. = R.H.S. = 0 and $v = 0u$

If $v \neq 0$, consider the orthogonal decomposition

$$u = \frac{\langle u,v\rangle}{\|v\|^2} v + w \quad \text{where } v \perp w$$

By Pythagorean Theorem

$$\|u\|^2 = \left\| \frac{\langle u,v\rangle}{\|v\|^2} v \right\|^2 + \|w\|^2$$

$$= \frac{|\langle u,v\rangle|^2}{\|v\|^4} \|v\|^2 + \|w\|^2$$

$$\geq \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

$$\therefore \ \|u\|^2 \|v\|^2 \geq |\langle u,v\rangle|^2$$

Note equality holds $\iff w = \vec{0}$

$$\iff u = \frac{\langle u,v\rangle}{\|v\|^2} v$$

$\therefore$ Equality holds $\Rightarrow u = cv$ for some $c \in \mathbb{F}$

If $u = cv$, then

$$\frac{\langle u,v\rangle}{\|v\|^2} v = \frac{\langle cv,v\rangle}{\|v\|^2} v = cv = u$$

If $v = c$, then

$$\frac{\langle u,v\rangle}{\|v\|^2} v = \frac{\langle u,cu\rangle}{\|cu\|^2} cu = \frac{\bar{c}\|u\|^2}{|c|^2\|u\|^2} cu = u$$

$\therefore$ $u = cv$ or $v = cu \Rightarrow$ Equality holds

Prop 6.16 (Triangle Inequality)

$$\|u+v\| \le \|u\| + \|v\| \quad \text{for any } u, v \in V$$

Equality holds $\|u+v\| = \|u\| + \|v\|$ $\iff$ $u = cv$ or $v = cu$ for some $c \in \mathbb{R}$, $c \ge 0$

Pf

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v \rangle \\
&= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\
&= \|u\|^2 + \langle u,v\rangle + \overline{\langle u,v\rangle} + \|v\|^2 \\
&= \|u\|^2 + 2\,\mathrm{Re}\langle u,v\rangle + \|v\|^2 \\
&\le \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2 \\
&\le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \\
&= (\|u\| + \|v\|)^2
\end{aligned}$$

$\therefore \|u+v\| \le \|u\| + \|v\|$

Equality holds

$$\iff 2\,\mathrm{Re}\langle u,v\rangle \underset{①}{=} 2|\langle u,v\rangle| \underset{②}{=} 2\|u\|\|v\|$$

② Cauchy-Schwarz

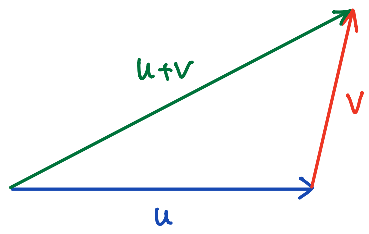
① $\iff \langle u,v\rangle \in \mathbb{R}, \ge 0$

② $\iff u = cv$ or $v = cu$ for some $c \in \mathbb{F}$

$\therefore$ Equality holds $\iff$ ①, ②

$\iff u = cv$ or $v = cu$ for some $c \in \mathbb{R}$, $c \ge 0$

Geometric Intepretation:



In a triangle, the length of any side is less than or equal to the sum of lengths of the other two sides